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THE STRESS ANALYSIS OF MULTI-LAYERED COMPOSITES WITH A FLAW†

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Abstract—The plane strain and anti-plane shear problems for the general multi-layered composites are considered. The main objective of the paper is to provide the theoretical tools for studying the fracture of layered composites with flaws. The problem of interest is, therefore, the determination of the disturbance in the stress state in the layered medium due to the existence of a flaw. A new method is developed to obtain the relevant system of integral equations for the problem. The system of equations is solved for some particular cases and the numerical results for the stress intensity factors are given.

1. INTRODUCTION

MULTI-LAYERED bonded plates and shells will perhaps be one of the most common and basic structural materials in the design and construction of aerospace and hydrospace vehicles in the coming decades. The basic appeal of the idea lies in the great flexibility it offers to design engineers, providing, for example, such properties as high modulus, yield and ultimate strength with high toughness through a combination of layers having various properties, directional strengthening of the structure by means of fiber or filament reinforced layers, obtaining a desirable damping characteristic by including a suitably selected visco-elastic layer, etc.

In studying the mechanical response of the layered composites, generally one may differentiate two groups of problems: the first relates to the bulk response of the composite and usually consists of problems concerning the determination of the mechanical properties of and the overall stress distribution in the medium which is assumed to be free from local imperfections. The second group of problems concerns the micromechanics of the medium, in which one is particularly interested in the response of the medium in the neighborhood of localized imperfections. Common forms of these imperfections are broken bonds on the interfaces, and voids, inclusions and dislocations in the layers. For mathematical convenience, these imperfections may all be classified as singular surfaces across which the displacement or the stress vector suffer a discontinuity. The importance of these problems lies in their application to the fracture of the composites, for it is reasonable to expect (and the practical evidence indicates) that these imperfections will generally form the nucleus of the fracture initiation and propagation in the medium.

In this paper, we will consider the plane strain and anti-plane shear problems for a medium composed of homogeneous, isotropic layers with different mechanical properties. After outlining the general procedure for obtaining the integral equations of the problem, we will concentrate on the specific examples concerning an elastic layer with a crack bonded

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to two elastic half planes. Even though the technique described here is quite general and the effects of additional layers, anisotropy and the interaction of more than one flaw may be taken into account in a straightforward manner, with the inclusion of these effects the analysis as well as the solution of the resulting integral equations become increasingly complicated, time consuming and costly.

The main interest of the paper is in the evaluation of the disturbance in the stress state due to the existence of a crack. Thus, assuming that the overall stress distribution in the imperfection-free medium is known, through a simple superposition the singular portion of the problem may be formulated as a stress disturbance problem in which the only external loads acting on the medium are the tractions on the crack surfaces.

The basic mathematical techniques used in the solution of the problem considered in this paper have been described in [1–3]. Solutions for some special cases dealing with the plane and anti-plane problems for bonded semi-infinite planes are given in [4–8].

2. GENERAL PROCEDURE FOR THE PLANE STRAIN PROBLEM

Consider the plane strain problem for the multi-layered medium shown in Fig. 1. The medium consists of $n+m$ layers $S_{-m}, \dots, S_n$ with different thicknesses and elastic properties. For simplicity let the medium contain only a single cut along $y = 0, |x| < 1$, and the surfaces of the cut be subjected to known tractions

$$\begin{aligned} \sigma_{yy}^{1}(x, 0) &= \sigma_{yy}^{-1}(x, 0) = p_1(x), \\ \sigma_{xy}^{1}(x, 0) &= \sigma_{xy}^{-1}(x, 0) = p_2(x) \end{aligned} \tag{1}$$

where the functions p_1 and p_2 satisfy a Hölder condition in $(-1, 1)$. Since p_1 and p_2 are the only external loads and since the medium possesses a geometric symmetry with respect to $x = 0$ plane, the problem may be treated as the sum of a symmetric and an anti-symmetric parts to be solved for $x \geq 0$. In this paper, we will further restrict our attention to the symmetric part of the problem in which we have

$$p_1(x) = p_1(-x), \qquad p_2(x) = -p_2(-x), \qquad |x| < 1. \tag{2}$$

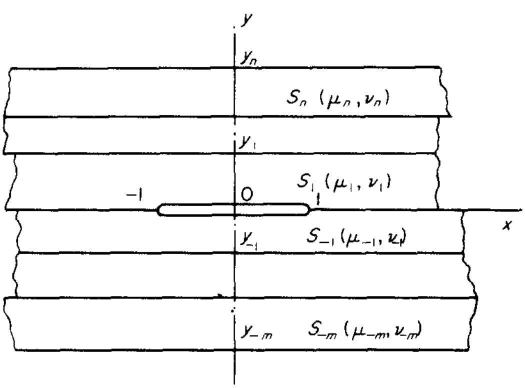

Fig. 1. Notation for bonded multi-layered medium.

The solution of the anti-symmetric part requires only a slight modification.

Assuming the x, y-components of the displacements in the ith layer in the form†

$$
\begin{aligned}
u_i(x, y) &= \frac{2}{\pi}\int_0^\infty \phi_i(\alpha, y)\sin\alpha x\,\mathrm{d}\alpha \\
v_i(x, y) &= \frac{2}{\pi}\int_0^\infty \psi_i(\alpha, y)\cos\alpha x\,\mathrm{d}\alpha, \qquad (i = -m, \ldots, -1, 1, \ldots, n)
\end{aligned}
\tag{3}
$$

and using the field equations

$$
\begin{aligned}
\mu_i\nabla^2 u_i + (\lambda_i+\mu_i)\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right) &= 0 \\
\mu_i\nabla^2 v_i + (\lambda_i+\mu_i)\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x}+\frac{\partial v_i}{\partial y}\right) &= 0
\end{aligned}
\tag{4}
$$

we obtain

$$
\begin{aligned}
u_i(x, y) &= \frac{2}{\pi}\int_0^\infty [(A_{i1}+A_{i2}y)\,\mathrm{e}^{-\alpha y}+(A_{i3}+A_{i4}y)\mathrm{e}^{\alpha y}]\sin\alpha x\,\mathrm{d}\alpha \\
v_i(x, y) &= \frac{2}{\pi}\int_0^\infty \left\{\left[A_{i1}+\left(\frac{\kappa_i}{\alpha}+y\right)A_{i2}\right]\mathrm{e}^{-\alpha y}+\left[-A_{i3}+\left(\frac{\kappa_i}{\alpha}-y\right)A_{i4}\right]\mathrm{e}^{\alpha y}\right\}\cos\alpha x\,\mathrm{d}\alpha
\end{aligned}
\tag{5}
$$

where $\kappa_i = 3-4\nu_i$ and A_{ij} are functions of α which will be determined from the boundary and the continuity conditions, and μ_i and ν_i are, respectively, the shear modulus and the Poisson's ratio. After obtaining u_i, v_i, the stresses may be evaluated by Hooke's Law. In particular, the components of the stress vector at the interfaces and boundaries may be expressed as,

$$
\begin{aligned}
\frac{1}{2\mu_i}\sigma^i_{yy} &= \frac{2}{\pi}\int_0^\infty \{-[\alpha(A_{i1}+A_{i2}y)+2(1-\nu_i)A_{i2}]\,\mathrm{e}^{-\alpha y} \\
&\quad +[-\alpha(A_{i3}+A_{i4}y)+2(1-\nu_i)A_{i4}]\,\mathrm{e}^{\alpha y}\}\cos\alpha x\,\mathrm{d}\alpha \\
\frac{1}{2\mu_i}\sigma^i_{xy} &= \frac{2}{\pi}\int_0^\infty \{-[\alpha(A_{i1}+A_{i2}y)+(1-2\nu_i)A_{i2}]\,\mathrm{e}^{-\alpha y} \\
&\quad +[\alpha(A_{i3}+A_{i4}y)-(1-2\nu_1)A_{i4}]\,\mathrm{e}^{\alpha y}\}\sin\alpha x\,\mathrm{d}\alpha.
\end{aligned}
\tag{6}
$$

On the boundaries $y = y_n$, $y = y_{-m}$, $0 \le x < \infty$ the medium may have formally any one of the following four groups of homogeneous boundary conditions:

$$
\begin{aligned}
&\text{(a)}\ \sigma^i_{yy} = 0 = \sigma^i_{xy}; \qquad \text{(b)}\ \ u_i = 0 = v_i; \\
&\text{(c)}\ \sigma^i_{xy} = 0 = v_i; \qquad \ \text{(d)}\ \sigma^i_{yy} = 0 = u_i; \qquad (i = -m, n).
\end{aligned}
\tag{7}
$$

† For the problems under consideration the external loads, p_1, p_2, are statically self-equilibrating. As a result, the displacements as well as their derivatives decrease sufficiently rapidly as $|x| \to \infty$ so that the requirement of absolute integrability is satisfied and they may be expressed as Fourier integrals.

The continuity conditions require that on the interfaces the stress and displacement vectors in the adjacent layers be equal, i.e.

$$\begin{aligned} u_{i+1}-u_i &= 0, \qquad & v_{i+1}-v_i &= 0 \\ \sigma_{yy}^{i+1}-\sigma_{yy}^{i} &= 0, \qquad & \sigma_{xy}^{i+1}-\sigma_{xy}^{i} &= 0 \end{aligned} \tag{8}$$

$$(i = -m, \dots, -2, y = y_{i+1}; \qquad i = 1, \dots, n, y = y_i; 0 \le x < \infty).$$

Now, to obtain the integral equations for the problem, we will first assume that at $y = 0$ the bond between the two adjacent layers is perfect except for the (symmetrically located) dislocations at $y = 0$, $x = t$ defined by †

$$\frac{\partial}{\partial x}(u_1-u_{-1}) = f_1\delta(x-t), \qquad \frac{\partial}{\partial x}(v_1-v_{-1}) = f_2\delta(x-t) \qquad (y = 0) \tag{9}$$

where f_1, f_2 are constants and (9) constitutes the only "external load" acting on the medium. In addition to (9), on the interface $y = 0$ we have the conditions

$$\sigma_{yy}^{1}-\sigma_{yy}^{-1} = 0, \qquad \sigma_{xy}^{1}-\sigma_{xy}^{-1} = 0 \qquad (0 \le x < \infty, y = 0). \tag{10}$$

Conditions (7)–(10) provide $4(n+m)$ equations to determine the same number of unknown functions $A_{ik}(\alpha)$. By substituting from (5) and (6) into (7), (8) and (10) and taking the inverse transforms, we obtain $4(n+m)-2$ linear homogeneous algebraic equations in A_{ik}. Again, substituting from (5) into (9) and inverting we find

$$\begin{aligned} &\alpha(A_{11}+A_{13}-A_{-11}-A_{-13}) = f_1 \cos \alpha t, \\ &-\alpha(A_{11}-A_{13}-A_{-11}+A_{-13})-\kappa_1(A_{12}+A_{14})+\kappa_{-1}(A_{-12}+A_{-14}) = f_2 \sin \alpha t. \end{aligned} \tag{11}$$

Thus, with (11), the system of equations obtained from (7), (8) and (10) may be solved for A_{ij} in the following general form:

$$A_{ij}(\alpha) = F_{ij}^1(\alpha) f_1 \cos \alpha t + F_{ij}^2(\alpha) f_2 \sin \alpha t \qquad (i = -m, \dots, -1, 1, \dots, n; j = 1, \dots, 4) \tag{12}$$

where the functions $F_{ij}^k(\alpha)$ depend on the material constants and the geometry of the medium.

Substituting (12) into (6), the two stress components of interest in the layer S_1 due to the dislocations (9) may be expressed as

$$\begin{aligned} \frac{1}{2\mu_1}\sigma_{yy}^1(x, y, t) &= h_{11}(x, y, t) f_1 + h_{12}(x, y, t) f_2 \\ \frac{1}{2\mu_1}\sigma_{xy}^1(x, y, t) &= h_{21}(x, y, t) f_1 + h_{22}(x, y, t) f_2. \end{aligned} \tag{13}$$

The functions h_{ij} are given in terms of infinite integrals involving $F_{ij}^k(\alpha)$.

† Here in the analysis we are considering the half plane $x > 0$. In order to introduce the dislocations, the cut is made between 0 and t. Thus the relative displacements are constant for $0 < x < t$ and zero for $x > t$, i.e.

$$u_1-u_{-1} = f_1[H(x-t)-1], \; v_1-v_{-1} = f_2[H(x-t)-1]$$

where $H(\;)$ is the Heaviside step function. The crack opening is formed through the superposition of these relative displacements. Note that the integrability conditions of displacements are still satisfied.

Considering the nature of the input (9), (13) may now be considered as the Green's functions for the problem. That is, instead of (9), if we have the x-derivatives of the relative displacements given as functions of $t, f_1(t), f_2(t)$, along a certain portion L of the interface $y = 0$, the stresses in the layer S_1 may be obtained by simply replacing f_i in (13) by $f_i(t)$ and integrating in t along L. In particular, if the cut L extends from -1 to $+1$ and if the tractions along the cut are known to be $p_1(x)$, $p_2(x)$ [see: (1)], then taking again into consideration the symmetry, from (13) we obtain

$$\lim_{y\to 0}\int_0^1 \sum_1^2 h_{ij}(x, y, t) f_j(t)\,\mathrm{d}t = \frac{1}{2\mu_1} p_i(x) \qquad (i = 1, 2), (0 \le x < 1). \tag{14}$$

For the physical problem under consideration, the stresses on the crack surface, $p_i(x)$, are known and the displacement derivatives, $f_i(t)$ are unknown, which may be determined from the integral equations given by (14). Since Green's functions of the form (13) may easily be written for all the desired field quantities, for the solution of the problem it may then be sufficient to determine the functions $f_i(t)$.

As will be seen from the specific examples considered in this paper, two of the kernels in (14) have Cauchy-type singularity, and the system of integral equations is of first kind if the crack is imbedded in a homogeneous layer and of second kind if the crack is located on the interface of two dissimilar materials. For the solution of the integral equations, it will be more convenient to express (14) in the range $-1 < x < 1$, $-1 < t < 1$ by using the symmetry properties of the functions f_i and p_i, $(i = 1, 2)$.

Here it should be clearly noted that in writing (14) from the Green's functions (13) we have used the condition that $f_j(t) = 0$, $(j = 1, 2)$ for $|t| > 1$. However, on $y = 0$, $|x| > 1$, in addition to the continuity of the stress vector which has been used in deriving (13), the conditions which must be satisfied are, $u_1 - u_{-1} = 0$ and $v_1 - v_{-1} = 0$. Referring now to the definition of f_1, f_2, i.e.

$$f_1(x) = \frac{\partial}{\partial x}(u_1 - u_{-1}), \qquad f_2(x) = \frac{\partial}{\partial x}(v_1 - v_{-1}),$$

for the continuity of the displacements (aside from a rigid body translation) on $y = 0$, $|x| > 1$, f_1 and f_2 must satisfy the following conditions

$$\int_{-1}^{1} f_1(x)\,\mathrm{d}x = 0, \qquad \int_{-1}^{1} f_2(x)\,\mathrm{d}x = 0. \tag{15}$$

3. ANTI-PLANE SHEAR PROBLEMS FOR A MULTI-LAYERED MEDIUM

Again, consider the medium shown in Fig. 1 in which we now assume that all the external loads act in z direction. Thus, to solve the problem, it will be sufficient to determine the z-components w_i of the displacement vector in the layers S_i. Restricting our attention again to symmetric problems, and expressing w_i in terms of the Fourier integrals

$$w_i(x, y) = \frac{2}{\pi}\int_0^\infty \theta_i(y, \alpha)\cos\alpha x\,\mathrm{d}\alpha \qquad (i = -m, \ldots, -1, 1, \ldots, n), \tag{16}$$

from the field equations

$$\nabla^2 w_i(x, y) = 0 \tag{17}$$

we obtain

$$w_i(x, y) = \frac{2}{\pi} \int_0^\infty (A_i\, e^{\alpha y} + B_i\, e^{-\alpha y}) \cos \alpha x \, d\alpha. \tag{18}$$

The nonvanishing stress components are

$$\begin{aligned} \sigma_{xz}^i(x, y) &= \mu_i \frac{\partial w_i}{\partial x} \\ \sigma_{yz}^i(x, y) &= \mu_i \frac{\partial w_i}{\partial y} = \mu_i \frac{2}{\pi} \int_0^\infty \alpha(A_i\, e^{\alpha y} - B_i\, e^{-\alpha y}) \cos \alpha x \, d\alpha. \end{aligned} \tag{19}$$

Following the procedure outlined in the previous section, from (18) and (19) the boundary conditions at $y = y_n$, $y = y_{-m}$, and the continuity conditions across the interfaces with no cuts may be written as

$$\begin{aligned} &\mu_n \alpha(A_n - B_n\, e^{-2\alpha y_m}) = 0 \\ &\mu_{-m} \alpha(A_{-m}\, e^{2\alpha y_{-m}} - B_{-m}) = 0 \\ &(A_{i+1} + B_{i+1}\, e^{-2\alpha y_i}) - (A_i + B_i\, e^{-2\alpha y_i}) = 0 \qquad (i = 1, \ldots, n-1) \\ &(A_j\, e^{2\alpha y_j} + B_j) - (A_{j-1}\, e^{2\alpha y_j} + B_{j-1}) = 0 \qquad (j = -m+1, \ldots, -1) \\ &\mu_{i+1}(A_{i+1} - B_{i+1}\, e^{-2\alpha y_i}) - \mu_i(A_i - B_i\, e^{-2\alpha y_i}) = 0 \qquad (i = 1, \ldots, n-1) \\ &\mu_j(A_j\, e^{2\alpha y_j} - B_j) - \mu_{j-1}(A_{j-1}\, e^{2\alpha y_j} - B_{j-1}) = 0 \qquad (j = -m+1, \ldots, -1). \end{aligned} \tag{20}$$

Instead of stress boundary conditions assumed in (20) (first two equations), we may have displacement or stress and displacement boundary conditions. Again, assuming a dislocation of strength f at $y = 0$, $x = t$ defined by

$$\frac{\partial}{\partial x}(w_1 - w_{-1}) = f\delta(x-t), \qquad (y = 0)$$

to be the only external load, for the conditions at $y = 0$ we have

$$\begin{aligned} \mu_1(A_1 - B_1) - \mu_{-1}(A_{-1} - B_{-1}) &= 0 \\ -\alpha(A_1 + B_1) + \alpha(A_{-1} + B_{-1}) &= f \sin \alpha t. \end{aligned} \tag{21}$$

The unknown functions A_i, B_i may be determined from (20) and (21). In particular, by simple elimination, it is easy to show that

$$\begin{aligned} A_1(\alpha) &= G_1(\alpha) B_1(\alpha), \\ \alpha B_1(\alpha) &= -\frac{\mu_{-1}}{\mu_1 + \mu_{-1}} [1 + G_2(\alpha)] f \sin \alpha t \\ G_1(\alpha) &= 0(e^{-\alpha y_1}), \qquad G_2(\alpha) = 0(e^{-\alpha y_1}, e^{\alpha y_{-1}}) \quad \text{for} \quad \alpha \to \infty. \end{aligned} \tag{22}$$

The stress component σ_{yz}^1 due to the dislocation f may then be expressed as

$$\sigma_{yz}^1(x, y, t) = \frac{\mu_1 \mu_{-1}}{\mu_1 + \mu_{-1}} f \frac{2}{\pi} \int_0^\infty (1 + G_2)(e^{-\alpha y} - G_1\, e^{\alpha y}) \sin \alpha t \cos \alpha x \, d\alpha. \tag{23}$$

If there is a crack on the interface $y = 0, |x| < 1$, acted upon by tractions

$$\sigma^1_{yz}(x, 0) = \sigma^{-1}_{yz}(x, 0) = p(x) = p(-x), \qquad |x| < 1 \tag{24}$$

f may be assumed to be a function of t which is zero for $|t| > 1$, and is unknown in $|t| < 1$. Thus using the Green's function (23), the integral equation for f may be expressed as

$$\frac{\mu_1+\mu_{-1}}{\mu_1\mu_{-1}}p(x) = \lim_{y\to 0^+}\int_0^1 f(t)\,\mathrm{d}t\frac{2}{\pi}\int_0^\infty \mathrm{e}^{-\alpha y}\sin\alpha t\cos\alpha x\,\mathrm{d}\alpha$$
$$+\int_0^1 f(t)\,\mathrm{d}t\frac{2}{\pi}\int_0^\infty (G_2-G_1G_2-G_1)\sin\alpha t\cos\alpha x\,\mathrm{d}\alpha \tag{25}$$

where, because of uniform convergence [see third equation of (22)], in the second term the limit has been put under the integral sign. At this point we require that, because of the boundedness and smoothness of the crack opening displacement $w_1 - w_{-1}$, the unknown function f be continuous in $0 \le x < 1$ and be integrable around $x = 1$. Under these conditions and using $f(t) = -f(-t)$, the first term in the right hand side of (25) becomes [2]

$$\lim_{y\to 0^+}\frac{2}{\pi}\int_0^1 f(t)\,\mathrm{d}t\int_0^\infty \frac{1}{2}\mathrm{e}^{-\alpha y}[\sin\alpha(t-x)+\sin\alpha(t+x)]\,\mathrm{d}\alpha$$
$$= \lim_{y\to 0^+}\frac{1}{\pi}\int_{-1}^1 f(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-\alpha y}\sin\alpha(t-x)\,\mathrm{d}\alpha$$
$$= \frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{t-x}\,\mathrm{d}t. \tag{26}$$

Similarly, expressing the second term in $(-1, 1)$, (25) may be written as

$$\frac{\mu_1+\mu_{-1}}{\mu_1\mu_{-1}}p(x) = \frac{1}{\pi}\int_{-1}^1 \frac{f(t)\,\mathrm{d}t}{t-x} + \int_{-1}^1 k(x, t)f(t)\,\mathrm{d}t, \qquad |x| < 1 \tag{27}$$

where the Fredholm kernel $k(x, t)$ is given by

$$k(x, t) = \frac{1}{\pi}\int_0^\infty (G_2-G_1G_2-G_1)\sin\alpha(t-x)\,\mathrm{d}\alpha. \tag{28}$$

Note that $k(x, t)$ goes to zero as $y_1 \to \infty$, $y_{-1} \to -\infty$ and we recover the simple integral equation for the bonded half planes [8].

Again, in deriving (25) we used the condition $(\partial/\partial x)(w_1 - w_{-1}) = f(x) = 0$, rather than $w_1 - w_{-1} = 0$, for $|x| > 1$, $y = 0$. Thus the continuity of the displacements requires that (27) must be solved under the following single-valuedness condition:

$$\int_{-1}^1 f(t)\,\mathrm{d}t = 0. \tag{29}$$

4. EXAMPLES

In this section we will consider various specific examples of some practical interest. The main problem which will be studied is that of an elastic layer bonded to two dissimilar half planes. It will be assumed that the layer contains a through crack parallel to the interfaces and is subjected to in-plane or anti-plane surface tractions on the crack surfaces.

4.1 *Anti-plane shear of an elastic layer bonded to two half planes*

The geometry of the first example is shown by the insert in Fig. 2. In this case, noting that $y_2 = \infty, y_{-2} = -\infty, \mu_1 = \mu_{-1}$ (see Fig. 1), if we solve the equations (20)–(22) with the notation of Fig. 2, we obtain

$$
\begin{aligned}
G_2(\alpha) &= \lambda_2 \, e^{-2\alpha h_2}, \\
G_1(\alpha) &= -\lambda_1(\lambda_2 \, e^{-2\alpha h} + e^{-2\alpha h_1})/(1-\lambda_1\lambda_2 \, e^{-2\alpha h}) \\
\lambda_1 &= (\mu_1-\mu_3)/(\mu_1+\mu_3), \ \lambda_2 = (\mu_2-\mu_3)/(\mu_2+\mu_3).
\end{aligned}
\tag{30}
$$

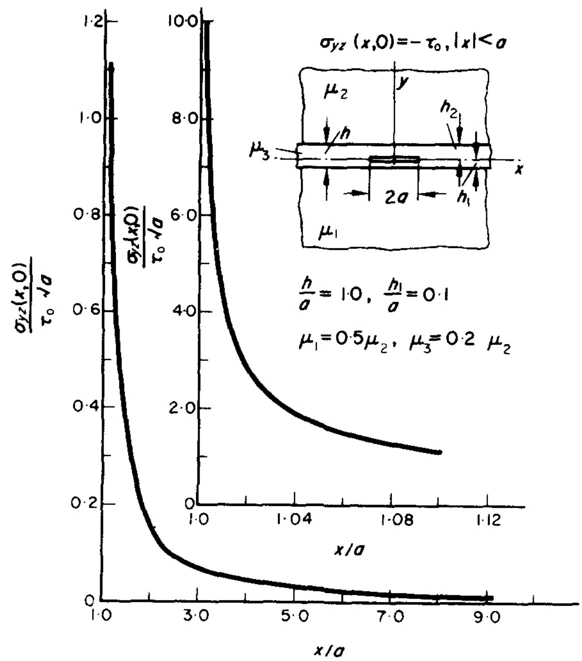

FIG. 2. Stress distribution along $y = 0, x > 1$ for the anti-plane case.

Thus, once the traction $p(x)$ is specified the integral equation (27) gives the solution by using the kernel obtained from (28) and (30) and the condition (29). Singular integral equations of the form (27) has been very extensively studied (see, for example, [1]). Equation (27) may be solved by reducing it to an integral equation with a weakly singular kernel to which the Fredholm theorems are applicable. This reduction may be accomplished by using either the singular adjoint operator to regularize the integral equation, or the method of Carleman and Vekua in which the integral containing $k(x, t)$ is considered part of the input and the method of solution of the dominant system is followed.

Considering the dominant part of (27), the fundamental function of the singular equation may easily be determined as follows [1]:

$$R(t) = (1-t)^{\frac{1}{2}+n_1}(1+t)^{-\frac{1}{2}+n_2} \tag{31}$$

where n_1 and n_2 are arbitrary integers which are determined through physical considerations. After deformation since the crack opening displacement around $|x| \leq 1$ is parabolic rather than a cusp, at $x = \mp 1$ the displacement derivative $f(x)$ must have a singularity. And also for $|x| \leq 1$ since the displacement is finite, the singularity of f must be integrable. Aside from a multiplying analytic function of t, the behavior of f around $|x| = 1$ is entirely determined by the fundamental function $R(x)$. Thus, it is easy to see that, for the problem under consideration the constants in (31) are $n_1 = -1, n_2 = 0$ and R becomes†

$$R(t) = (1-t^2)^{-\frac{1}{2}}. \tag{32}$$

The index of the singular integral equation (27) is $\kappa = -(n_1+n_2) = 1$, meaning that the solution of (27) will be determinate within an arbitrary constant, which is determined from the condition (29).

To solve (27), instead of using the standard methods mentioned above which are numerically very cumbersome, we will use the method described in [3]. This method is based on the basic observation that the fundamental function $R(t)$ of the integral equation is the weight function of Chebishev polynomials of the first kind. Thus expressing the unknown function as

$$\frac{\mu_3}{2} f(t) = R(t) \sum_{0}^{\infty} A_n T_{2n+1}(t) \tag{33}$$

and using the relations

$$\int_{-1}^{1} T_j(t)(1-t^2)^{-\frac{1}{2}} \frac{\mathrm{d}t}{t-x} = \begin{cases} 0, & j = 0 \\ \pi U_{j-1}(x), & j > 0 \end{cases} \tag{34}$$

the singularity of the equation may be removed and the problem may be reduced to an infinite system of linear algebraic equations in the unknown coefficients A_n. In (34) $U_j(x)$ is the Chebishev polynomial of the second kind. Note that with the choice of f as an odd function in (33), the single-valuedness condition (29) is automatically satisfied. Thus substituting from (33) into (27) we obtain

$$p(x) = \sum_{0}^{\infty} [U_{2j}(x) + H_{2j+1}(x)] A_j \tag{35}$$

where

$$H_{2j+1}(x) = \int_{-1}^{1} k(x,t) T_{2j+1}(t)(1-t^2)^{-\frac{1}{2}}\, \mathrm{d}t. \tag{36}$$

† From the related Hilbert problem it is easy to show that in this problem the end points ∓ 1 are non-special ends. Hence, at these points the function is either zero or unbounded (see [1], §79). This information, which is necessary to determine the index of the integral equation, is basically physical and cannot be deducted from the mathematics of the problem. In considering the solution of singular integral equations, if the unknown function is a "flux"-type quantity (e.g. stress, displacement derivative, heat flux, velocity) the solution sought is of the class h_0 and the function has integrable singularities at all non-special ends. On the other hand if the unknown function is a "potential"-type quantity (e.g. displacement, temperature, velocity potential) the solution is of the class h_m (m being the number of the non-special ends) and the function is bounded at all ends. Since in most applications the unknown function falls in the former category and since h_0 is the most general class of solutions, in some applications the lengthy physical considerations are replaced by a mathematical "hypothesis". However, the h_0 assumption (or hypothesis) may not always lead to the most convenient solution (see, for instance, the example in §99 of [1] on the rigid stamp with rounded corners) or may not even be correct, as, for example in the case of elastostatics of cracked shells where the structure of the integral equations is identical to that of (60) but the unknown functions are essentially in-plane displacements [11], hence the fundamental solution is of class h_m rather than h_0.

The algebraic equations are obtained from (35) by using a weighted residual method with the orthogonal system $U_i(x)$ as the weight functions. Defining the constants

$$F_n = \int_{-1}^{1} p(x)U_{2n}(x)(1-x^2)^{\frac{1}{2}}\,dx, \qquad (n = 0, 1, \ldots)$$
$$a_{nj} = \int_{-1}^{1} H_{2j+1}(x)U_{2n}(x)(1-x^2)^{\frac{1}{2}}\,dx, \qquad (n, j = 0, 1, \ldots) \tag{37}$$

and using the orthogonality relation

$$\int_{-1}^{1} U_n(x)U_j(x)(1-x^2)^{\frac{1}{2}}\,dx = \begin{cases} 0, & n \neq j \\ \pi/2, & n = j \end{cases} \tag{38}$$

we then obtain

$$F_n = \frac{\pi}{2}A_n + \sum_{0}^{\infty} a_{nj}A_j, \qquad (n = 0, 1, \ldots). \tag{39}$$

In the problems considered in this paper the systems of equations such as (39) are solved by using the method of reduction, and since the convergence was very satisfactory in all cases, no more than twelve unknown coefficients (or pairs of coefficients) were needed for a three significant digit accuracy in the stress intensity factors for the worst combination of dimensions, (i.e. the smallest values of h_1, h_2 or h used in the calculations). The number of unknown coefficients was adjusted for the desired degree of accuracy in the stress intensity factors.

By examining the behavior of the general solution in the close neighborhood of the crack tip (see, for example [8]), the stress intensity factor defined by

$$k_3 = \lim_{x \to 1} \sqrt{(x^2-1)}\sigma_{yz}(x, 0) \tag{40}$$

may be related to the derivative of the crack opening displacement, $f(x)$ as follows

$$k_3 = -\lim_{x \to 1} \frac{\mu_1\mu_{-1}}{\mu_1+\mu_{-1}}\sqrt{(1-x^2)}f(x). \tag{41}$$

Since $\mu_1 = \mu_{-1} = \mu_3$, from (41) and (33) we obtain

$$k_3 = -\sum_{0}^{\infty} A_n. \tag{42}$$

For a constant shear $\sigma_{yz}(x, 0) = -\tau_0$ and material properties $\mu_1 = 0{\cdot}5\mu_2$, $\mu_3 = 0{\cdot}2\mu_2$ the results are shown in Figs. 2–4. Figure 2 shows the distribution of the shear stress for $x > 1$ on the plane of the crack for $h_1/h = 0{\cdot}1$. Figure 3 shows the stress intensity factor k_3 as a function of h_1/h for various values h/a. Figure 3 indicates that, because of the stiffer half-planes bonded to its sides, the stress intensity factor in the layer is always smaller than the value $\tau_0\sqrt{a}$ corresponding to the infinite medium. Also, it is seen that as the relative thickness of the layer h/a decreases, the stress intensity factor too decreases. This is also seen in Fig. 4 which shows the variation of k_3 (along with some plane strain results) for $h_1 = h/2$ and the material properties given in Fig. 2 as a function of h/a.† Figure 3

† Note the difference in definition of h in Figs. 3 and 4.

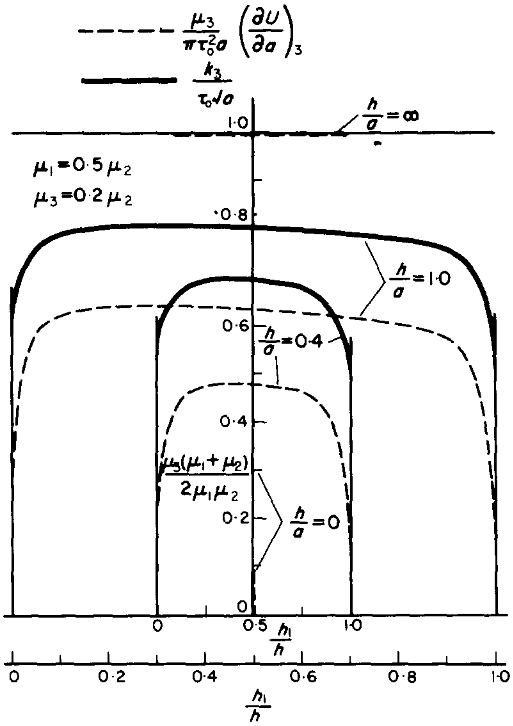

FIG. 3. Stress intensity factor for the anti-plane case.

also shows that as the crack approaches the interfaces, there is again a sharp decrease in k_3. Finally we note that if the stiffness of the layer is greater than that of the half planes, the above trends concerning the magnitude of k_3 will be reversed.

In the problems of layered composites, as the crack approaches an interface, part of the Fredholm kernel increases very rapidly and, as a result, the numerical analysis requires more care. In limit for $h_1 = 0$ or $h_2 = 0$ part of the Fredholm kernel becomes a Cauchy kernel which has to be separated and combined with the first term in (27). For example in (30), if $h_1 = 0$ it may easily be shown that (27) becomes

$$\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{t-x}+\int_{-1}^{1}k(x,t)f(t)\,\mathrm{d}t = \frac{\mu_1+\mu_3}{\mu_1\mu_3}p(x), \qquad |x|<1 \\ &k(x,t) = \frac{1}{\pi}\int_{0}^{\infty}\frac{(1+\lambda_1)\lambda_2\,\mathrm{e}^{-2\alpha h}}{1-\lambda_1\lambda_2\,\mathrm{e}^{-2\alpha h}}\sin\alpha(t-x)\,\mathrm{d}\alpha. \end{aligned} \tag{43}$$

Further, if we also let $h \to 0$ in (43), $k(x, t)$ becomes a Cauchy kernel and (43) reduces to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{t-x} = \frac{\mu_1+\mu_2}{\mu_1\mu_2}p(x), \qquad |x|<1$$

which is the integral equation for two bonded half planes with an interface crack [8].

Figure 3 also shows the variation of the strain energy release rate as a function of h_1/h for various values of h/a. In the notation of Fig. 1, this quantity may be obtained from

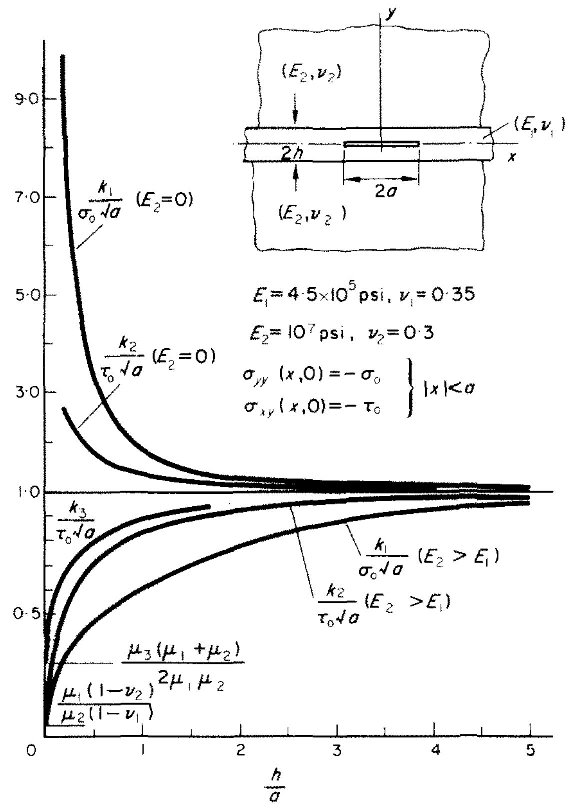

FIG. 4. Stress intensity factors for the plane strain case.

the stresses and displacements around the crack tips as the rate of closure energy for a crack propagating in its own plane as follows:

$$\left(\frac{\partial U}{\partial a}\right)_{-1,1} \mathrm{d}a = 2\int_a^{a+\mathrm{d}a} \tfrac{1}{2}\sigma_{yz}(x)[w_1(x-\mathrm{d}a)-w_{-1}(x-\mathrm{d}a)]\,\mathrm{d}x$$

$$= \frac{\pi(\mu_1+\mu_{-1})}{2\mu_1\mu_{-1}}(k_3)^2_{-1,1}\,\mathrm{d}a$$

or, for the example under consideration,

$$\frac{\mu_3}{\pi}\left(\frac{\partial U}{\partial a}\right)_3 = (k_3)_3^2, \qquad 0 < \frac{h_1}{h} < 1,$$

$$\frac{\mu_3}{\pi}\left(\frac{\partial U}{\partial a}\right)_{3,1} = 0{\cdot}7(k_3)_{3,1}^2, \qquad h_1 = 0,$$

$$\frac{\mu_3}{\pi}\left(\frac{\partial U}{\partial a}\right)_{3,2} = 0{\cdot}6(k_3)_{3,2}^2, \qquad h_1 = h.$$

It is seen that as the crack approaches to an interface with a stiffer material there is even a sharper decrease in $\partial U/\partial a$ (compared with that in k_3). Since $\partial U/\partial a$ represents the rate of energy available for fracture, h_1 corresponding to $(\partial U/\partial a)$ = maximum would give the fracture plane, provided the shear cleavage resistance of the bonds is sufficiently high. For example, if γ denotes the shear cleavage strength, for $h/a = 0{\cdot}4$ fracture would take place within the layer 3 provided $\gamma_{1,3} > 0{\cdot}48\gamma_3$ and $\gamma_{2,3} > 0{\cdot}33\gamma_3$.

4.2 *Plane strain problem for an elastic layer bonded to two identical half planes*

As a second example we consider the plane strain problem of an elastic layer bonded to two half planes of the same elastic properties. In this relatively simple example it will be assumed that the crack is located in the mid-section of the layer (see the insert in Fig. 4). Thus, under properly symmetric tractions on the crack surfaces, $y = 0$ and $x = 0$ planes are planes of symmetry for the problem, and it is sufficient to consider the quadrant $x \geq 0$, $y \geq 0$ only. Referring to (5) and the insert in Fig. 4 it is seen that $A_{23} = 0 = A_{24}$ and the remaining six functions $A_{11}, \ldots, A_{14}, A_{21}, A_{22}$ are determined from the conditions at $y = h$ and $y = 0$. At $y = h$ the continuity of displacements and stresses require that

$$u_1 = u_2, \qquad v_1 = v_2, \qquad \sigma^1_{yy} = \sigma^2_{yy}, \qquad \sigma^1_{xy} = \sigma^2_{xy}, \qquad y = h, \qquad x \geq 0. \tag{44}$$

For the symmetric problem at $y = 0$ we have

$$\sigma^1_{xy} = 0, \qquad y = 0, \qquad x \geq 0. \tag{45}$$

Substituting from (5) and (6) into (44) and (45) we obtain five algebraic equations in $A_{ij}(\alpha)$. The sixth equation is obtained by considering a dislocation as $x = t$ as follows:

$$\frac{\partial v_1}{\partial x} = \tfrac{1}{2} f_2 \delta(x-t), \qquad y = 0, \tag{46}$$

from which, using (5), we obtain

$$-\alpha(A_{11}-A_{13}) - \kappa_1(A_{12}+A_{14}) = \tfrac{1}{2} f_2 \sin \alpha t. \tag{47}$$

After obtaining $A_{ij}(\alpha)$ if we substitute them into (6) we obtain the Green's functions for the stresses. In particular the normal stress at $y = 0$ may be obtained as

$$\begin{aligned}
\sigma^1_{yy}(x, 0, t) &= \frac{2\mu_1}{1+\kappa_1} \lim_{y\to 0^+} \frac{2}{\pi} f_2 \int_0^\infty [\mathrm{e}^{-\alpha y} + G_1(\alpha)] \sin \alpha t \cos \alpha x \, \mathrm{d}\alpha \\
G_1(\alpha) &= \frac{(1 - a_1 a_2 + 4\alpha h + 4\alpha^2 h^2 - 2a_1 \,\mathrm{e}^{-2\alpha h})\,\mathrm{e}^{-2\alpha h}}{a_2 - 4\alpha h \,\mathrm{e}^{-2\alpha h} + a_1 \,\mathrm{e}^{-4\alpha h}} \\
a_1 &= \frac{\mu_1\kappa_2 - \mu_2\kappa_1}{\mu_1\kappa_2 + \mu_2}, \qquad a_2 = \frac{\mu_2\kappa_1 + \mu_1}{\mu_2 - \mu_1}.
\end{aligned} \tag{48}$$

If we assume that there is a cut in the layer along $|x| < 1$, $y = 0$, and the surfaces of the cut are subjected to tractions given by

$$\sigma^1_{yy}(x, 0) = p_1(x) = p_1(-x), \qquad \sigma^1_{xy}(x, 0) = 0, \qquad |x| < 1, \tag{49}$$

f_2 may now be considered as a function of t which is zero for $|t| > 1$ and unknown for $|t| < 1$. Through physical considerations identical to that of the previous example, we further require that at $x = \mp 1$ the function $f_2(x) = 2(\partial/\partial x)v_1(x, 0)$ be discontinuous and have an integrable singularity. From (48) and (49) the integral equation for $f_2(t)$ may then be obtained as

$$\frac{1+\kappa_1}{2\mu_1}p_1(x) = \lim_{y\to 0^+}\frac{2}{\pi}\int_0^1 f_2(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-\alpha y}\sin\alpha t\cos\alpha x\,\mathrm{d}\alpha$$
$$+\frac{2}{\pi}\int_0^1 f_2(t)\,\mathrm{d}t\int_0^\infty G_1(\alpha)\sin\alpha t\cos\alpha x\,\mathrm{d}\alpha. \tag{50}$$

Noting that f_2 is an odd function and following the same procedure as that of the previous example [see: equations (25)–(27)], (50) may be expressed as

$$\frac{1}{\pi}\int_{-1}^1\frac{f_2(t)\,\mathrm{d}t}{t-x}+\int_{-1}^1 k_1(x,t)f_2(t)\,\mathrm{d}t = \frac{1+\kappa_1}{2\mu_1}p_1(x), \qquad |x|<1$$
$$k_1(x,t) = \frac{1}{\pi}\int_0^\infty G_1(\alpha)\sin\alpha(t-x)\,\mathrm{d}\alpha \tag{51}$$

where the function G_1 is given by (48). Again, the condition of single-valuedness of the displacement v_1 requires that the function $f_2(t)$ must satisfy the following relation:

$$\int_{-1}^1 f_2(t)\,\mathrm{d}t = 0. \tag{52}$$

For the anti-symmetric problem if we assume that the tractions on the crack surfaces are

$$\sigma_{yy}^1 = 0, \qquad \sigma_{xy}^1 = p_2(x) = p_2(-x), \qquad |x|<1, \qquad y=0$$

and the dislocation f_1 is defined as

$$\tfrac{1}{2}f_1(x) = \frac{\partial u_1}{\partial x}, \qquad |x|<1, \qquad y=0 \tag{53}$$

it may be shown that the integral equation (51) remains valid with f_1, k_2 and p_2 respectively replacing f_2, k_1 and p_1, the only difference being in the Fredholm kernel which, for this case, may be expressed as

$$k_2(x,t) = \frac{1}{\pi}\int_0^\infty\frac{(1-a_1a_2-4\alpha h+4\alpha^2h^2-2a_1\,\mathrm{e}^{-2\alpha h})\,\mathrm{e}^{-2\alpha h}}{a_2+4\alpha h\,\mathrm{e}^{-2\alpha h}+a_1\,\mathrm{e}^{-4\alpha h}}\sin\alpha(t-x)\,\mathrm{d}\alpha. \tag{54}$$

In plane strain crack problems the symmetric and anti-symmetric components of the stress intensity factors may be defined as

$$k_1 = \lim_{x\to 1}\sqrt{(x^2-1)}\sigma_{yy}(x,0)$$
$$k_2 = \lim_{x\to 1}\sqrt{(x^2-1)}\sigma_{xy}(x,0). \tag{55}$$

By examining the behavior of the stresses and the crack surface displacements in the close neighborhood of the crack tip for the general solution (see, e.g. [1]), the constants k_1, k_2 may be related to the functions f_1, f_2 as follows:

$$k_1 = -\frac{2\mu}{1+\kappa}\lim_{x\to 1}\sqrt{(1-x^2)}f_2(x)$$
$$k_2 = -\frac{2\mu}{1+\kappa}\lim_{x\to 1}\sqrt{(1-x^2)}f_1(x). \tag{56}$$

The results obtained from the solution of (51) by using the tractions $p_1(x) = -\sigma_0$ or $p_2(x) = -\tau_0$ for symmetric and anti-symmetric problems, respectively, are shown in Fig. 4. The two cases considered in Fig. 4 are $E_2 = 0$ (i.e. the layer with stress-free boundaries) and $E_2 > E_1$. As expected, in the first case the stress intensity factors k_1, k_2 are greater than the values of $\sigma_0\sqrt{a}$ and $\tau_0\sqrt{a}$ corresponding to the infinite homogeneous plane, whereas in the second case they are smaller. The figure shows that if $E_2 > E_1$, the fracture resistance of the layer increases as h/a ratio decreases. On the other hand, for $E_2 = 0$, $k_i \to \infty$ as $h/a \to 0$ $(i = 1, 2)$.

Again, from (51), (48) and (54) it may easily be verified that for $h \to 0$ the kernels $k_i(x, t)$ become Cauchy kernels, and after combining with the first terms, for the symmetric and the anti-symmetric cases (51) respectively reduces to

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^{1}\frac{f_2(t)\,dt}{t-x} &= \frac{1+\kappa_2}{2\mu_2}p_1(x), \qquad |x| < 1 \\ \frac{1}{\pi}\int_{-1}^{1}\frac{f_1(t)\,dt}{t-x} &= \frac{1+\kappa_2}{2\mu_2}p_2(x), \qquad |x| < 1 \end{aligned} \tag{57}$$

which are the integral equations for the homogeneous plane with elastic properties μ_2, κ_2. Note that in (57) the unknown functions f_i are

$$f_2(x) = \frac{\partial}{\partial x}(v_2^+ - v_2^-), \qquad f_1(x) = \frac{\partial}{\partial x}(u_2^+ - u_2^-), \qquad y = 0$$

whereas for $h \neq 0$, i.e. in (51), they are

$$f_2(x) = \frac{\partial}{\partial x}(v_1^+ - v_1^-), \qquad f_1(x) = \frac{\partial}{\partial x}(u_1^+ - u_1^-), \qquad y = 0.$$

In going to limit $h = 0$ the quantities which remain continuous are the crack opening displacements, i.e.

$$(u_1^+ - u_1^-) \to (u_2^+ - u_2^-), \qquad (v_1^+ - v_1^-) \to (v_2^+ - v_2^-).$$

Thus for $a = 1$ and $h \to 0$ we may write

$$\begin{aligned} -(k_1 + ik_2)_1\frac{1+\kappa_1}{2\mu_1} &= \lim_{x\to 1}\sqrt{(1-x^2)}(f_2 + if_1)_1 \to \lim_{x\to 1}\sqrt{(1-x^2)}(f_2+if_1)_2 \\ &= -(k_1+ik_2)_2\frac{1+\kappa_2}{2\mu_2} = -\frac{1+\kappa_2}{2\mu_2}(\sigma_0 + i\tau_0) \end{aligned} \tag{58}$$

or

$$(k_1 + ik_2)_1 \to \frac{\mu_1(1-\nu_2)}{\mu_2(1-\nu_1)}(\sigma_0 + i\tau_0) \tag{59}$$

giving the limiting values of the stress intensity factors. In (58) and (59) the second subscripts refer to the materials 1 and 2 shown in Fig. 4.

For the anti-plane shear problem the limiting value of k_3 shown in Fig. 4 was obtained in a similar way. However, the values of k_3 shown in Fig. 3 for $(h_1/h) = 0$ and 1 had to be obtained by solving the integral equations corresponding to interface cracks, in which the coefficient of p in (27) was $(\mu_1+\mu_3)/\mu_1\mu_3$ for $h_1 = 0$ and $(\mu_2+\mu_3)/\mu_2\mu_3$ for $h_1 = h$, and the Fredholm kernels $k(x, t)$ were different.

4.3 *Elastic layer bonded to dissimilar half planes*

As a third and somewhat more interesting example we consider the analysis of a cracked layer bonded to two half planes with different elastic constants (see the insert in Fig. 6). The physical problem is that of bonded half planes in which the thickness of the bonding agent is not negligible and the bonding material contains an imperfection which may be idealized as a crack. In this problem defining

$$\begin{aligned}\lambda_1 &= (\kappa_1\mu_3-\kappa_3\mu_1)/(\mu_1+\kappa_1\mu_3),\\ \lambda_2 &= (\kappa_2\mu_3-\kappa_3\mu_1)/(\mu_2+\kappa_2\mu_3),\\ \lambda_3 &= (\mu_3+\kappa_3\mu_1)/(\mu_1-\mu_3),\\ \lambda_4 &= (\mu_3+\kappa_3\mu_2)/(\mu_2-\mu_3)\end{aligned}$$

and following the procedure described in the previous sections, after rather lengthy manipulations, the integral equations for the unknown functions f_1, f_2 and the corresponding Fredholm kernels may be expressed as

$$\begin{aligned}&\frac{1}{\pi}\int_{-1}^{1}\frac{f_2(t)\,\mathrm{d}t}{t-x}+\int_{-1}^{1}\sum_{1}^{2}k_{1j}(x,t)f_j(t)\,\mathrm{d}t = \frac{1+\kappa_3}{2\mu_3}p_1(x),\\ &\frac{1}{\pi}\int_{-1}^{1}\frac{f_1(t)\,\mathrm{d}t}{t-x}+\int_{-1}^{1}\sum_{1}^{2}k_{2j}(x,t)f_j(t)\,\mathrm{d}t = \frac{1+\kappa_3}{2\mu_3}p_2(x), \qquad |x|<1,\end{aligned} \tag{60}$$

$$\begin{aligned}k_{11}(x,t) = -\frac{1}{2\pi}\int_0^\infty &\Big\{\lambda_3L_3(\alpha)+(2\alpha h_1-1)L_1(\alpha)\\ &+\left[\lambda_1L_1(\alpha)-\lambda_1\frac{1-4\alpha^2h_1^2}{\lambda_3}+(1+2\alpha h_1)L_3(\alpha)\right]\mathrm{e}^{-2\alpha h_1}\Big\}\cos\alpha(t-x)\,\mathrm{d}\alpha,\\ k_{12}(x,t) = -\frac{1}{2\pi}\int_0^\infty &\Big\{\lambda_3L_4(\alpha)+(2\alpha h_1-1)L_2(\alpha)\\ &+\left[\lambda_1L_2(\alpha)+\lambda_1\frac{(1+2\alpha h_1)^2}{\lambda_3}+(1+2\alpha h_1)L_4(\alpha)\right]\mathrm{e}^{-2\alpha h_1}\Big\}\sin\alpha(t-x)\,\mathrm{d}\alpha,\\ k_{21}(x,t) = \frac{1}{2\pi}\int_0^\infty &\Big\{-\lambda_3L_3(\alpha)-(2\alpha h_1+1)L_1(\alpha)\\ &+\left[\lambda_1L_1(\alpha)-\lambda_1+\frac{(2\alpha h_1-1)^2}{\lambda_3}+(2\alpha h_1-1)L_3(\alpha)\right]\mathrm{e}^{-2\alpha h_1}\Big\}\sin\alpha(t-x)\,\mathrm{d}\alpha,\\ k_{22}(x,t) = -\frac{1}{2\pi}\int_0^\infty &\Big\{-\lambda_3L_4(\alpha)-(2\alpha h_1+1)L_2(\alpha)\\ &+\left[\lambda_1L_2(\alpha)+\lambda_1+\frac{1-4\alpha^2h_1^2}{\lambda_3}+(2\alpha h_1-1)L_4(\alpha)\right]\mathrm{e}^{-2\alpha h_1}\Big\}\cos\alpha(t-x)\,\mathrm{d}\alpha\end{aligned}$$

where

$$L_1(\alpha) = (F_2+F_1)/F_6, \qquad L_2(\alpha) = (F_3-F_1)/F_6,$$

$$L_3(\alpha) = \left(F_4+\frac{1-2\alpha h_1}{\lambda_3}F_1\right) \Big/ F_6, \qquad L_4(\alpha) = \left(F_5+\frac{1+2\alpha h_1}{\lambda_3}F_1\right) \Big/ F_6,$$

$$F_1(\alpha) = (\lambda_4+G_5-G_3)[G_2-(\lambda_4+G_5)\,\mathrm{e}^{-2\alpha h_1}]-G_4(2\alpha h_1\lambda_4+G_1)+\lambda_3\lambda_4(G_5-G_3),$$

$$F_2(\alpha) = (\lambda_4+G_5)[G_4-G_2+(\lambda_4+G_5)\,\mathrm{e}^{-2\alpha h_1}]-\lambda_3\lambda_4G_5,$$

$$F_3(\alpha) = (\lambda_4+G_5)[G_4+G_2-(\lambda_4+G_5)\,\mathrm{e}^{-2\alpha h_1}]+\lambda_3\lambda_4G_5,$$

$$F_4(\alpha) = (\lambda_4+G_5)(G_1+G_3-G_5)+\lambda_4G_5(2\alpha h_1-1),$$

$$F_5(\alpha) = (\lambda_4+G_5)(G_3-G_5-G_1)-\lambda_4G_5(2\alpha h_1+1),$$

$$F_6(\alpha) = \lambda_3\lambda_4^2+F_1,$$

$$G_1(\alpha) = 2\alpha h\,\mathrm{e}^{-2\alpha h_2}-2\alpha h_2\lambda_1\,\mathrm{e}^{-2\alpha h},$$

$$G_2(\alpha) = \lambda_3\,\mathrm{e}^{-2\alpha h_2}+\lambda_4\,\mathrm{e}^{-2\alpha h_1}+(1-4\alpha^2h_1h_2)\,\mathrm{e}^{-2\alpha h},$$

$$G_3(\alpha) = (1-4\alpha^2h_1h_2)\,\mathrm{e}^{-2\alpha h_2}-\lambda_2\,\mathrm{e}^{-4\alpha h_2}-\lambda_1\,\mathrm{e}^{-2\alpha h}+\lambda_1\lambda_2\,\mathrm{e}^{-2\alpha(h+h_2)},$$

$$G_4(\alpha) = 2\alpha h\,\mathrm{e}^{-2\alpha h}-2\alpha h_1\lambda_2\,\mathrm{e}^{-2\alpha(h+h_2)}+2\alpha h_2\lambda_3\,\mathrm{e}^{-2\alpha h_2},$$

$$G_5(\alpha) = (1-\lambda_2\lambda_4+4\alpha^2h_2^2)\,\mathrm{e}^{-2\alpha h_2}-\lambda_2\,\mathrm{e}^{-4\alpha h_2}.$$

In deriving (60) it is again assumed that for $|x| > 1$, $y = 0$, $f_1 = 0 = f_2$ rather than $u_3^+ - u_3^- = 0 = v_3^+ - v_3^-$. Thus (60) must be solved under the following single-valuedness conditions

$$\int_{-1}^{1} f_1(t)\,\mathrm{d}t = 0, \qquad \int_{-1}^{1} f_2(t)\,\mathrm{d}t = 0. \tag{61}$$

In the expressions given here μ_i is the shear modulus and $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for generalized plane stress. The equations given above may be reduced to those of the previous example in a straightforward manner by letting $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$. This has been verified analytically as well as numerically. Also, for $h_1 \to \infty$, $h_2 \to \infty$ it is easy to see that all the Fredholm kernels, k_{ij}, go to zero, and (60) reduces to the uncoupled system for a homogeneous plane with a crack [see: (57)].

For the problems in this section we will only consider the following external loads:

$$\sigma_{yy}^3(x, 0) = -\sigma_0, \qquad \sigma_{xy}^3(x, 0) = 0, \qquad |x| < 1. \tag{62}$$

Because of the symmetry in the medium with respect to $x = 0$ plane, under the tractions (62) the functions f_i will have the following symmetry properties:

$$f_1(t) = f_1(-t), \qquad f_2(t) = -f_2(-t).$$

Examining the dominant part of (60) and using the same arguments as that used in Section 4.1, it is easy to see that the fundamental functions for both f_1 and f_2 are

$R(t) = (1-t^2)^{-\frac{1}{2}}$. Thus to solve the system of singular integral equations (60) we let

$$\begin{aligned} \frac{2\mu_3}{1+\kappa_3} f_1(t) &= (1-t^2)^{-\frac{1}{2}} \sum_0^\infty A_{2n} T_{2n}(t) \\ \frac{2\mu_3}{1+\kappa_3} f_2(t) &= (1-t^2)^{-\frac{1}{2}} \sum_0^\infty A_{2n+1} T_{2n+1}(t) \end{aligned} \tag{63}$$

from which the stress intensity factors may be obtained as

$$k_1 = -\sum_0^\infty A_{2n+1}, \qquad k_2 = -\sum_0^\infty A_{2n}. \tag{64}$$

As an application we will first consider the case in which the half planes 1 and 2 have the same elastic constants with the crack at an arbitrary location in the layer. For $h_1 = h_2$ the problem was considered in the previous example. For $h_1 \neq h_2$, since the symmetry with respect to $y = 0$ plane no longer exists, $f_1(x) \neq 0$ for $|x| < 1$. Hence $\sigma_{xy} \neq 0$ for $|x| > 1$ and k_2 is nonzero, becoming increasingly more significant as the crack is moved closer to one of the interfaces. Figure 5 shows the results for one case in which the thickness of the layer h is equal to the total crack length $2a$ and the elastic constants are the same as that shown in Fig. 4.

If the layer is a brittle material the fracture initiation at the crack tips would be expected to take place perpendicular to the direction of maximum cleavage stress $\sigma_{\theta\theta}$, which, in

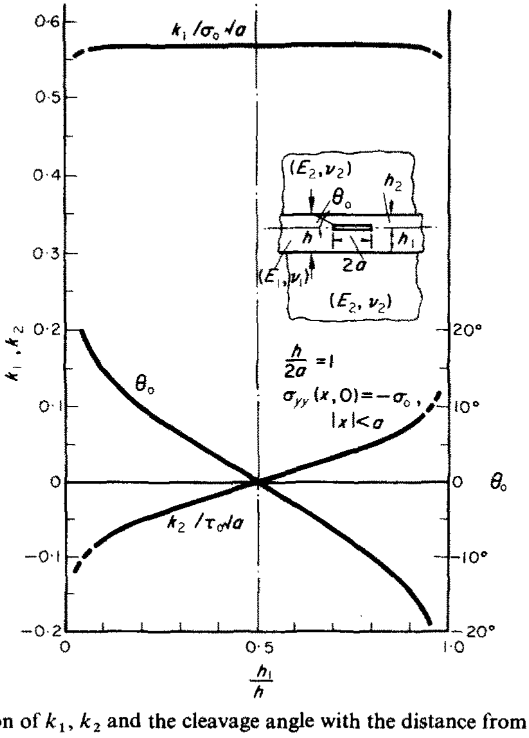

FIG. 5. Variation of k_1, k_2 and the cleavage angle with the distance from the interface.

the close neighborhood of the crack tip, may be expressed as [9],

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2r}}\cos\frac{\theta}{2}\left[k_1\cos^2\frac{\theta}{2} - \frac{3}{2}k_2\sin\theta\right] + O(\sqrt{r}). \tag{65}$$

In the particular example under consideration for $0{\cdot}05 < h_1/h < 0{\cdot}95\sqrt{(2r)}$, $\sigma_{\theta\theta}$ is approximately constant.† On the other hand, as it can be seen from Fig. 5, the variation of the cleavage angle θ_0 with h_1/h may be quite significant. From Fig. 5 it is seen that, if the adjacent medium is stiffer (in this case $E_2 > E_1$), the crack tends to propagate away from the interface. Thus, it may be thought that in this case (quasistatic) crack propagation would take place in the mid-plane of the layer. However, theoretically for certain $h/2a$ ratios and $h_1/h = 0{\cdot}5$, $\sigma_{\theta\theta}$ is a minimum. Hence, the crack propagation along the mid-plane is basically unstable. Taking into account also other possible imperfections (in load symmetry, geometry and material properties), it is more likely that the crack path would be a wavy curve with a long wave length which wanders between the neighborhoods of the two interfaces. If there are no imperfections in the bonds adhering the layer to the half planes, the crack path should not reach or intersect the interface.

For three different materials, Figs. 6 and 7 show some numerical results. The material constants used in the calculations and shown in Fig. 6 correspond to a steel–epoxy–aluminum combination. Figure 6 shows the variation of the stress intensity factors, k_1, k_2 as functions of h/a. In the results shown in Fig. 6 the crack is located in the mid-section of the layer. Note that in this example, in spite of the symmetry in loading and geometry, the shear component of the stress intensity factor k_2 and fracture initiation angle θ_0 are not zero.

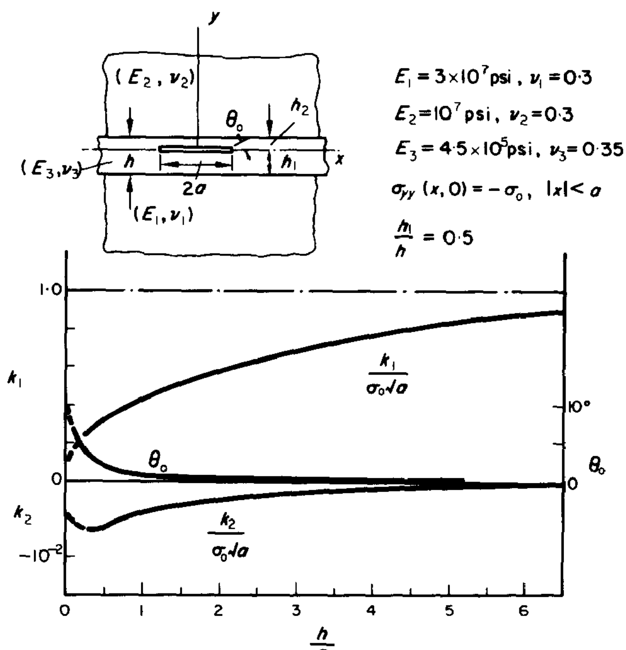

FIG. 6. Stress intensity factors for three different materials.

† In this example $\sqrt{(2r)}\sigma_{\theta\theta}$ goes through maxima around $h_1/h = 0{\cdot}1$ and $0{\cdot}9$ where it has a value of $1{\cdot}014\,k_1$.

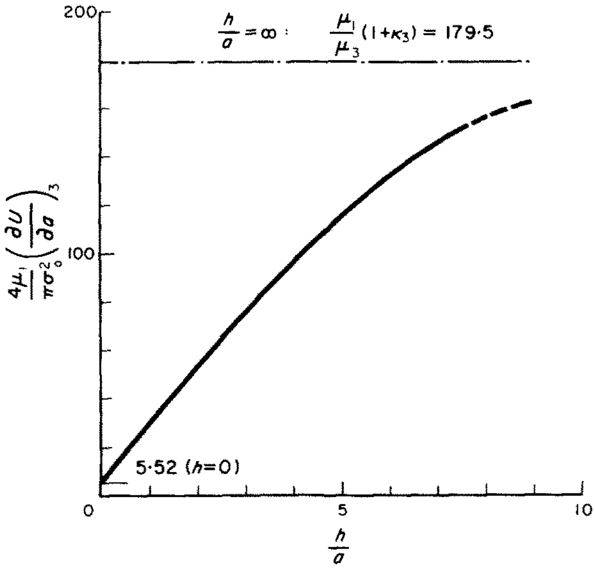

FIG. 7. Strain energy release rate for three different materials.

However, due to the relatively small difference between the moduli of steel and aluminum, k_2 is rather small compared with k_1.

For $h = 0$ the problem reduces to that of bonded dissimilar half planes with an interface crack for which the integral equations may be expressed as

$$\begin{aligned} \frac{p_1(x)}{\mu_2 b_2} &= -\gamma f_1(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{f_2(t)\,\mathrm{d}t}{t-x} \\ \frac{p_2(x)}{\mu_2 b_2} &= \frac{1}{\pi}\int_{-1}^{1}\frac{f_1(t)\,\mathrm{d}t}{t-x} + \gamma f_2(x) \\ \gamma &= b_1/b_2, \qquad |x| < 1 \\ b_1 &= \frac{\mu_1}{\mu_1+\kappa_1\mu_2} - \frac{\mu_1}{\mu_2+\kappa_2\mu_1}, \qquad b_2 = \frac{\mu_1}{\mu_1+\kappa_1\mu_2} + \frac{\mu_1}{\mu_2+\kappa_2\mu_1}. \end{aligned} \tag{66}$$

(66) can be solved in closed form [1] giving

$$\begin{aligned} f_1(t) + if_2(t) &= \frac{\sigma_0}{\mu_2 b_2(1+\gamma)}(t-2i\beta)R(t), \qquad |t| < 1 \\ \sigma_{yy}^2 - i\sigma_{xy}^2 &= \sigma_{yy}^1 - i\sigma_{xy}^1 = 2\mu_2 b_2 F(t) \\ &= \sigma_0[(t-2i\beta)R(t) - 1], \qquad |t| > 1 \\ R(t) &= \left(\frac{t+1}{t-1}\right)^{i\beta}(t^2-1)^{-\frac{1}{2}}, \; \beta = \frac{1}{2\pi}\log\left(\frac{1+\gamma}{1-\gamma}\right). \end{aligned} \tag{67}$$

Defining the stress intensity factor by

$$\begin{aligned} k_1 - ik_2 &= \lim_{t \to 1+0} (\sigma_{yy} - i\sigma_{xy}) \frac{1}{R(t)} \\ &= \lim_{t \to 1+0} (\sigma_{yy} - i\sigma_{xy}) \left(\frac{t-1}{t+1}\right)^{i\beta} \sqrt{(t^2-1)} \end{aligned} \tag{68}$$

we find

$$k_1 - ik_2 = \sigma_0(1 - 2i\beta). \tag{69}$$

Now expressing $f_1 + if_2$ around $t = 1$ and using (69) we may write

$$\begin{aligned} (k_1 - ik_2)_{1-2} &= \lim_{t \to 1-0} (f_1 + if_2) \frac{\mu_2 b_2(1+\gamma)}{R(t)} \\ &= \lim_{t \to 1-0} (f_1 + if_2)\mu_2 b_2(1+\gamma) i\sqrt{(1-t^2)} \sqrt{\left(\frac{1-\gamma}{1+\gamma}\right)} \left(\frac{1-t}{1+t}\right)^{i\beta} \\ &= -\lim_{t \to 1-0} \mu_2 b_2 (1-\gamma^2)^{\frac{1}{2}} \left(\frac{1-t}{1+t}\right)^{i\beta} \sqrt{(1-t^2)}(f_2 - if_1) \end{aligned} \tag{70}$$

which, for the homogeneous case (i.e. $\beta = 0$, $\gamma = 0$) reduces to (56).

From (56) and (70) it is seen that, because of the oscillation term $[(1-t)/(1+t)]^{i\beta}$, a limiting value of $(k_1 - ik_2)_3$ for $h \to 0$ cannot be obtained in terms of the known quantity $(k_1 - ik_2)_{1-2}$ by equating the crack opening displacements for the two cases. However, the quantity which is continuous and which can be computed in both cases is the rate of crack closure energy (generally known as the strain energy release rate for a slowly propagating crack). For the two cases $h \neq 0$ and $h = 0$ the closure energy for a crack propagating in its own plane may be expressed as [9, 10]

$$\begin{aligned} \left(\frac{\partial U}{\partial a}\right)_3 &= \frac{\pi(1+\kappa_3)}{4\mu_3}(k_1^2 + k_2^2)_3 \\ \left(\frac{\partial U}{\partial a}\right)_{1-2} &= \frac{\pi}{2} \frac{(\mu_1 + \kappa_1\mu_2)(\mu_2 + \kappa_2\mu_1)}{\mu_1\mu_2[(1+\kappa_1)\mu_2 + (1+\kappa_2)\mu_1]}(k_1^2 + k_2^2)_{1-2} \end{aligned} \tag{71}$$

where, $(k_1 - ik_2)_{1-2}$ is given by (69). $(\partial U/\partial a)_3$ is a function of h and as $h \to 0$ $(\partial U/\partial a)_3 \to (\partial U/\partial a)_{1-2}$.

For the elastic constants shown in Fig. 6 we find

$$\frac{4\mu_1}{\pi\sigma_0^2}\left(\frac{\partial U}{\partial a}\right)_{1-2} = 5{\cdot}52. \tag{72}$$

The variation of $(\partial U/\partial a)_3$ with h/a and its limiting value for $h \to 0$ given by (72) are shown in Fig. 7, which also shows the limiting value of $(\partial U/\partial a)_3$ as $h \to \infty$.

For a fixed value of layer thickness-to-crack length ratio (in this case, unity), the variation of k_1 and k_2 with the relative position of the crack is shown in Fig. 8. The results shown in Figs. 5 and 8 are quite similar. In Fig. 5 both half planes are aluminum. In Fig. 8 the lower half plane is replaced by steel which has a larger modulus. As a result, there is a slight reduction in the magnitudes of k_1 and k_2. Their distribution and the distribution of the predicted angle of fracture initiation θ_0 in h_1/h have no longer the symmetry properties observed in Fig. 5.

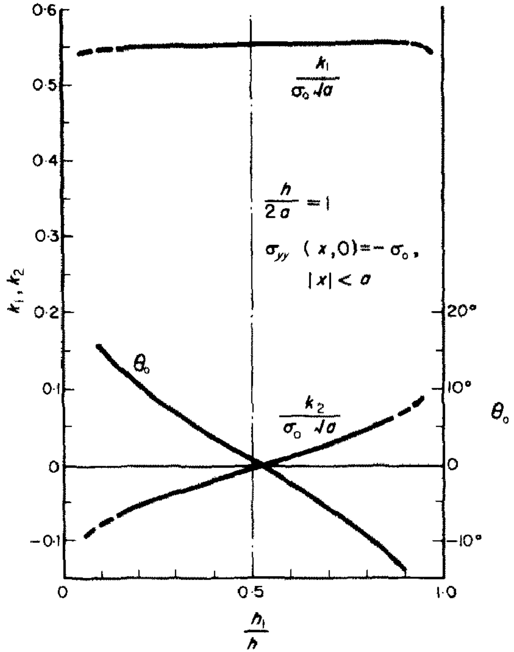

FIG. 8. Variation of stress intensity factors with respect to crack location in three different materials.

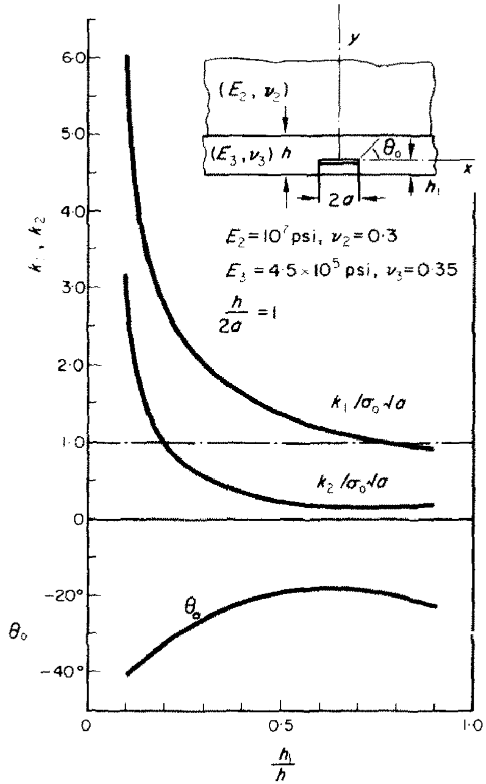

FIG. 9. Stress intensity factors for the elastic layer bonded to an elastic half-plane.

Finally, as a special case of the foregoing example we consider the problem of an elastic layer bonded to an elastic half-plane where the layer contains a crack parallel to its free surface (see the insert in Fig. 9). For the case of uniform normal tractions $\sigma_{yy}^3 = -\sigma_0$ applied to the crack surface, a fixed $h/2a$ ratio, and a material combination of aluminum–epoxy the numerical results are shown in Fig. 9. The figure shows that generally, the stress intensity factors increase indefinitely as the crack approaches the free surface and decrease as the distance from the crack to the interface decreases. However, as seen from the figure, for the dimensions and the material combination considered here the shear component of the stress intensity factor k_2 goes through a slight minimum around $h_1/h = 0{\cdot}75$. Physically this result is not unexpected. The minimum would be more pronounced for larger values of relative layer thickness and would disappear for smaller $h/2a$ ratios. Figure 9 also shows the predicted angle of fracture initiation θ_0 at the crack tips.

REFERENCES

[1] N. I. Muskhelishvili, *Singular Integral Equations*. Noordhoff (1953).
[2] F. Erdogan, Simultaneous dual integral equations with trigonometric and Bessel kernels. *Z. angew. Math. Mech.* **48**, 217 (1968).
[3] F. Erdogan, Approximate solution of systems of singular integral equations. *J. appl. Math.* **17**, 1041 (1969).
[4] F. Erdogan, Stress distribution in a nonhomogeneous elastic plane with cracks. *J. appl. Mech.* **30**, Series E, 232 (1963).
[5] F. Erdogan, Stress distribution in bonded dissimilar materials with cracks. *J. appl. Mech.* **32**, Series E, 403 (1965).
[6] A. H. England, A crack between dissimilar media. *J. appl. Mech.* **32**, Series E, 400 (1965).
[7] J. R. Rice and G. C. Sih, Plane problems of cracks in dissimilar media. *J. appl. Mech.* **32**, 418 (1965).
[8] F. Erdogan, Elastic–plastic anti-plane problems for bonded dissimilar media containing cracks and cavities. *Int. J. Solids Struct.* **2**, 447 (1966).
[9] F. Erdogan and G. C. Sih, On the crack extension in plates under plane loading and transverse shear. *J. bas. Engng* **85**, Series D, 519 (1963).
[10] B. M. Malyshev and R. L. Salganik, The strength of adhesive joints using the theory of fracture. *Int. J. Fracture Mech.* **1**, 114 (1965).
[11] F. Erdogan and J. J. Kibler, Cylindrical and spherical shells with cracks. *Int. J. Fracture Mech.* **5**, 229 (1969).

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Абстракт—Рассматриваются задачи плоской деформации и антиплоского сдвига для общих многослойных сложных структур. Глаыной целрю работы является сравнение теоретических рассуждений для исследования излома многослойных структур с трещинами. Затем, основным вопросом является опоределение возмущения напряженного состояния в слоистой среде, вследстие существования трещины. Выводится новый метод определения относящейся к данному вопросу системы интеглальных уравнений задачи. Решается система уравнений для некоторых частных случаев и даются численные разультаты для козффициентов интенсивности напряжений.